

Superconvergence of Immersed Finite Element Methods for Interface Problems

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Abstract

In this article, we study superconvergence properties of immersed finite element methods for the one dimensional elliptic interface problem. Due to low global regularity of the solution, classical superconvergence phenomenon for finite element methods disappears unless the discontinuity of the coefficient is resolved by partition. We show that immersed finite element solutions inherit all desired superconvergence properties from standard finite element methods without requiring the mesh to be aligned with the interface. In particular, on interface elements, superconvergence occurs at roots of generalized orthogonal polynomials that satisfy both orthogonality and interface jump conditions.

Keywords: superconvergence, immersed finite element method, interface problems, generalized orthogonal polynomials

1. Introduction

Immersed finite element (IFE) methods are a class of finite element methods (FEM) for solving differential equations with discontinuous coefficients, often known as interface problems. Unlike the classical FEM whose mesh is required to be aligned with the interface, IFE methods do not have such restriction on mesh. Consequently, IFE methods can use more structured,

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or even uniform meshes to solve interface problem regardless of interface location. This flexibility is advantageous for problems with complicated interfacial geometry [36] or for dynamic simulation involving a moving interface [21, 27, 28].

The main idea of IFE methods is to adapt approximating functions instead of meshes to fit the interface. On elements containing (part of) the interface, which we call interface elements, universal polynomials cannot approximate the solution accurately because of the low regularity of solution at the interface. A simple remedy is to construct piecewise polynomials as basis functions on interface elements in order to mimic the exact solution. The first IFE method was developed by Li [24] for solving the one-dimensional two-point boundary value problem. Piecewise linear shape functions were constructed on interface elements to incorporate the interface jump conditions. Following this idea, a family of quadratic IFE functions were introduced in [8]. Later in [1, 2], Adjerid and Lin extended the IFE approximation to arbitrary polynomial degree, and proved the optimal error estimates in the energy and the L^2 -norms. In the past decade, IFE methods have also been extensively studied for a variety of interface problems in two dimension [18, 25, 26, 30, 31, 32] and three dimension [22, 36].

There have been many studies in the mathematical theories for IFE methods, for example [2, 16, 20, 24, 29]. Most of theoretical analysis focuses on error estimation in Sobolev H^1 - and L^2 - norms, but very few literatures are concerned with the pointwise convergence. To the best of our knowledge, there is no systematic study on superconvergence phenomenon of IFE methods. Superconvergence theory for classical finite element methods [4, 17, 37] are invalid for IFE methods, unless the discontinuity of coefficient is resolved by solution meshes.

Superconvergence phenomena of FEM were discussed as early as 1967 by Zienkiewicz and Cheung [44]. Later, Douglas and Dupont in [17] proved that the p -th order C^0 finite element method to the two-point boundary value problem converges with rate $O(h^{2p})$ at nodal points. Since then the superconvergence behavior of FEM had been studied intensively. We refer to [5, 6, 14, 23, 35, 37] for an incomplete list of references. In the meantime, there also has been considerable interest in studying superconvergence for other numerical methods, for example, spectral and spectral collocation methods [41, 42, 43], finite volume methods [7, 11, 13, 15, 39], discontinuous Galerkin and local discontinuous Galerkin methods [3, 9, 10, 12, 19, 38, 40].

In this article, we focus on the conforming p -th degree IFE methods for the prototypical one-dimensional elliptic interface problem. There are two major contributions in this article. First, we present a novel approach

for developing IFE basis functions. The idea is completely different from classical approaches [1, 2], and the construction is based on the theory of orthogonal polynomials. Our new IFE bases accommodate interface jump conditions, and they satisfy certain orthogonality conditions which will be specified later. These basis functions can be explicitly constructed without solving linear systems. In an interface element, these IFE bases are either polynomials or piecewise polynomials, hence we call them *generalized orthogonal polynomials*.

Next, we analyze superconvergence properties of IFE methods. We will show that superconvergence phenomena occur at the roots of generalized orthogonal polynomials. To be more specific, the convergence rate of p -th degree IFE solutions is $O(h^{2p})$ at nodal points. The accuracy at nodes can be improved to *exact* if the elliptic operator has only the diffusion term. The IFE solution converge to the exact solution with rate $O(h^{p+2})$ at the roots of *generalized Lobatto polynomials*, and the convergence rate of derivatives is escalated to $O(h^{p+1})$ at the roots of *generalized Legendre polynomials*. All the results can be viewed as an extension from the classic result for FEM [17].

The rest of the paper is organized as follows. In Section 2, we present an explicit approach to construct IFE basis functions based on the framework of orthogonal polynomials. In Section 3, we study the superconvergence properties of IFE solutions for general elliptic interface problems. In Section 4, we report some numerical results. A few concluding remarks are presented in Section 5.

2. Generalized Orthogonal Polynomials

Let $\Omega = (a, b)$ be an open interval in \mathbb{R} . Assume that $\alpha \in \Omega$ is an interface point such that $\Omega^- = (a, \alpha)$ and $\Omega^+ = (\alpha, b)$. Consider the following one-dimensional elliptic interface problem

$$-(\beta u')' + \gamma u' + cu = f, \quad x \in \Omega^- \cup \Omega^+, \quad (2.1)$$

$$u(a) = u(b) = 0. \quad (2.2)$$

The diffusion coefficient β has a finite jump across the interface α . Without loss of generality, we assume that β is a piecewise constant defined by

$$\beta(x) = \begin{cases} \beta^-, & \text{if } x \in \Omega^-, \\ \beta^+, & \text{if } x \in \Omega^+, \end{cases} \quad (2.3)$$

where $\min\{\beta^+, \beta^-\} > 0$. The coefficients γ and c are assumed to be constants. At the interface α , the solution is assumed to satisfy the interface jump conditions

$$\llbracket u(\alpha) \rrbracket = 0, \quad \llbracket \beta u'(\alpha) \rrbracket = 0, \quad (2.4)$$

where $\llbracket v(\alpha) \rrbracket := v(\alpha^+) - v(\alpha^-)$.

2.1. Standard Orthogonal Polynomials

Consider an interface-independent partition of Ω :

$$a = x_0 < x_1 < \cdots < x_{k-1} < \alpha < x_k < \cdots < x_N = b. \quad (2.5)$$

Define a mesh $\mathcal{T}_h = \{\tau_i\}_{i=1}^N$, where $\tau_i = (x_{i-1}, x_i)$. Denoted by $h_i = x_i - x_{i-1}$ the element size, and by $h = \max\{h_i, i = 1, \dots, N\}$ the mesh size of \mathcal{T}_h . Note that the element τ_k contains α , hence we call τ_k the interface element and the rest of elements $\tau_i, i \neq k$ noninterface elements.

On noninterface elements, we adopt standard orthogonal polynomials as finite element basis functions. As usual, we construct the basis functions on the reference element $\tau = [-1, 1]$, then map them to each physical element τ_i by appropriate affine transformation. Let $P_n(\xi)$ be the Legendre polynomial of degree n on τ . Clearly they satisfy the following orthogonality

$$\int_{-1}^1 P_m(\xi) P_n(\xi) d\xi = \frac{2}{2n+1} \delta_{mn}. \quad (2.6)$$

Define $\{\psi_n\}$ to be the family of Lobatto polynomials on τ ,

$$\psi_0(\xi) = \frac{1-\xi}{2}, \quad \psi_1(\xi) = \frac{1+\xi}{2}, \quad \psi_n(\xi) = \int_{-1}^{\xi} P_{n-1}(t) dt, \quad n \geq 2. \quad (2.7)$$

2.2. Generalized Legendre Polynomials

On the interface element τ_k containing α , we construct a sequence of polynomials satisfying both orthogonality and interface jump conditions. Again, we map τ_k to the reference interval $\tau = [-1, 1]$ containing the reference interface point $\hat{\alpha}$. Let $\hat{\beta}(\xi) = \beta(x)$ such that $\hat{\beta}(\xi) = \beta^-$ on $\tau^- = (-1, \hat{\alpha})$ and $\hat{\beta}(\xi) = \beta^+$ on $\tau^+ = (\hat{\alpha}, 1)$.

Define a sequence of orthogonal polynomials $\{L_n\}$ with the weight function $w(\xi) = \frac{1}{\hat{\beta}(\xi)}$, i.e.,

$$(L_n, L_m)_w := \int_{-1}^1 w(\xi) L_n(\xi) L_m(\xi) d\xi = c_n \delta_{mn}, \quad (2.8)$$

where $c_n = \|L_n\|_w^2 = \sqrt{(L_n, L_n)_w}$. If we require $\{L_n\}$ to be monic polynomials, then they can be uniquely constructed via the following three-term recurrence formula ([34], Theorem 3.1):

Remark 2.1. *Let $\{L_n\}$ be the family of monic orthogonal polynomials satisfying (2.8). Then $\{L_n\}$ can be constructed as follows*

$$L_0(\xi) = 1, \quad L_1(\xi) = \xi - a_0, \quad (2.9)$$

$$L_{n+1}(\xi) = (\xi - a_n)L_n(\xi) - b_n L_{n-1}(\xi), \quad n \geq 1, \quad (2.10)$$

where

$$\begin{aligned} a_n &= \frac{(\xi L_n, L_n)_w}{(L_n, L_n)_w}, \quad n \geq 0 \\ b_n &= \frac{(L_n, L_n)_w}{(L_{n-1}, L_{n-1})_w}, \quad n \geq 1. \end{aligned}$$

The polynomials $\{L_n\}$ are generalized from standard Legendre polynomials $\{P_n\}$ by allowing the weight function to be discontinuous. Hence, we call $\{L_n\}$ the *generalized Legendre polynomials*.

2.3. Generalized Lobatto Polynomials

Next, we define a sequence of piecewise polynomial $\{\phi_n\}$ in a similar manner as (2.7)

$$\phi_0(\xi) = \begin{cases} \frac{(1-\hat{\alpha})\beta^- + (\hat{\alpha}-\xi)\beta^+}{(1-\hat{\alpha})\beta^- + (1+\hat{\alpha})\beta^+}, & \text{in } \tau^-, \\ \frac{(1-\xi)\beta^-}{(1-\hat{\alpha})\beta^- + (1+\hat{\alpha})\beta^+}, & \text{in } \tau^+. \end{cases} \quad (2.11)$$

$$\phi_1(\xi) = \begin{cases} \frac{(1+\xi)\beta^+}{(1-\hat{\alpha})\beta^- + (1+\hat{\alpha})\beta^+}, & \text{in } \tau^-, \\ \frac{(\xi-\hat{\alpha})\beta^- + (1+\hat{\alpha})\beta^+}{(1-\hat{\alpha})\beta^- + (1+\hat{\alpha})\beta^+}, & \text{in } \tau^+. \end{cases} \quad (2.12)$$

$$\phi_n(\xi) = \int_{-1}^{\xi} w(t) L_{n-1}(t) dt, \quad n = 2, 3, \dots \quad (2.13)$$

Note that ϕ_0 and ϕ_1 are constructed to satisfy nodal value conditions,

$$\phi_0(-1) = 1, \quad \phi_0(1) = 0, \quad \phi_1(-1) = 0, \quad \phi_1(1) = 1.$$

and the interface jump condition (2.4). In fact, ϕ_0 and ϕ_1 are piecewise linear polynomials, and they are two Lagrange type IFE nodal basis functions (see [2, 24]).

Theorem 2.1. $\{\phi_n\}$ is a sequence of piecewise polynomials and satisfy

- the interface jump conditions

$$\llbracket \phi_n(\hat{\alpha}) \rrbracket = 0, \quad \llbracket \hat{\beta} \phi'_n(\hat{\alpha}) \rrbracket = 0, \quad \forall n \geq 0, \quad (2.14)$$

- the weighted orthogonality condition

$$\langle \phi_m, \phi_n \rangle_{\hat{\beta}} := \int_{-1}^1 \hat{\beta}(\xi) \phi'_m(\xi) \phi'_n(\xi) d\xi = c_n \delta_{mn}, \quad \forall m, n \geq 1, \quad (2.15)$$

where c_n is some nonzero constant.

Proof. We first prove the interface jump conditions (2.14). It is true for ϕ_0 and ϕ_1 by direct verification using (2.11) and (2.12). For $n \geq 2$, we note that ϕ_n is a continuous function because it is defined through the integral (2.13). Moreover, since $\{L_n\}$ is a sequence of polynomials, then

$$\llbracket \hat{\beta} \phi'_n(\hat{\alpha}) \rrbracket = \beta^+ \phi'_n(\hat{\alpha}+) - \beta^- \phi'_n(\hat{\alpha}-) = L_{n-1}(\hat{\alpha}+) - L_{n-1}(\hat{\alpha}-) = 0.$$

The orthogonality (2.15) follows from (2.8) and (2.13), *i.e.*,

$$\langle \phi_m, \phi_n \rangle_{\hat{\beta}} = (L_{m-1}, L_{n-1})_w = \tilde{c}_{n-1} \delta_{m-1, n-1} = c_n \delta_{mn}.$$

□

The piecewise polynomials $\{\phi_n\}$ are generalized from standard Lobatto polynomials $\{\psi_n\}$ defined in (2.7). The construction (2.13) uses piecewise constant weight function $w(\xi) = \beta(\xi)^{-1}$ instead of universal constant one. We call $\{\phi_n\}$ the *generalized Lobatto polynomials* on the reference interface element τ .

The generalized Lobatto polynomials $\{\phi_n\}$ form a sequence of IFE basis functions satisfying both interface jump conditions and orthogonal conditions. In Figure 1, we plot a few generalized Legendre polynomials L_n and generalized Lobatto polynomials ϕ_n as an illustration. In these plots we choose $\hat{\alpha} = 0.15$, $\beta^- = 1$, and $\beta^+ = 5$.

Remark 2.2. The generalized Lobatto polynomials $\{\phi_n\}$ are identical (up to a multiple constant) to IFE basis functions introduced in [1]. However, the construction in this article is more explicit and does not require solving a linear system. This procedure is more advantageous when there are multiple discontinuities in an interval.

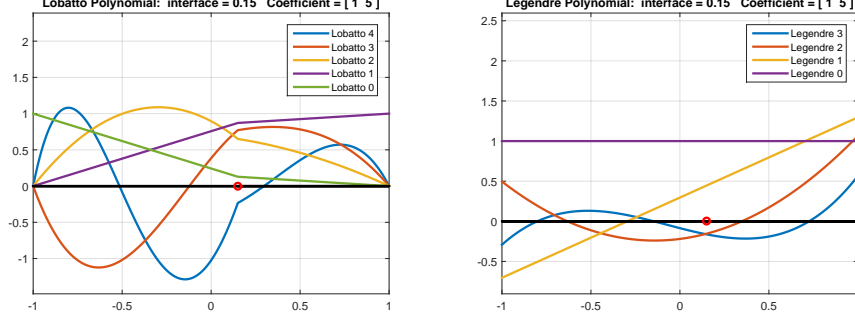


Figure 1: Generalized Lobatto (left) and Legendre (right) polynomials

Remark 2.3. *The construction of ϕ_n does not require the extended interface jump conditions [1]:*

$$\left[\beta \phi_n^{(j)}(\hat{\alpha}) \right] = 0, \quad \forall j = 2, 3, \dots, n. \quad (2.16)$$

However, it can be easily verified that all the generalized Lobatto polynomials $\{\phi_n\}$ satisfy (2.16) automatically.

2.4. Properties of Generalized Orthogonal Polynomials

In this subsection, we investigate some fundamental properties about the generalized orthogonal polynomials.

First, it is interesting to know the number and distribution of zeros for the generalized Lobatto polynomials and generalized Legendre polynomials in the interval $[-1, 1]$. To prove our main result, we need the following lemma.

Lemma 2.1. *(Generalized Rolle's theorem) Assume that f is real-valued and continuous on a closed interval $[a, b]$ with $f(a) = f(b)$. If for every x in the open interval (a, b) , both of one side limits*

$$f'(x+) = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}, \quad f'(x-) = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}$$

exist, then there is some number c in the open interval (a, b) such that one of the two limits $f'(c+)$ and $f'(c-)$ is ≥ 0 and the other is ≤ 0 .

The above lemma generalizes the Rolle's theorem to functions that are continuous on $[a, b]$, but not necessarily differentiable at all interior points of (a, b) . The proof is straightforward and similar to the standard Rolle's

theorem; hence we omit it in this article. Now we are ready to prove the main result of this section.

Theorem 2.2. *The generalized Legendre polynomials $\{L_n\}$ and generalized Lobatto polynomials $\{\phi_n\}$ have same numbers of roots as the standard Legendre polynomials $\{P_n\}$ and Lobatto polynomials $\{\psi_n\}$, respectively, i.e.,*

1. *For $n \geq 1$, L_n has n simple roots in the open interval $(-1, 1)$.*
2. *For $n \geq 1$, $\phi_{n+1}(\pm 1) = 0$, and ϕ_{n+1} has $n - 1$ simple “roots” in the open interval $(-1, 1)$, i.e., the piecewise polynomial $\phi_{n+1}(\xi)$ crosses the ξ -axis $n - 1$ times in $(-1, 1)$.*

Proof. Note that $\{L_n\}$ is a family of orthogonal polynomials on $[-1, 1]$. The weight function $w(\xi) = \beta(\xi)^{-1}$ is positive and is a Lebesgue integrable function. Hence, the polynomial L_n has n simple roots in $(-1, 1)$.

Now we consider the generalized Lobatto polynomial ϕ_{n+1} . By the definition (2.12), it is obvious that $\phi_{n+1}(-1) = 0$. By the orthogonality of L_n ,

$$\phi_{n+1}(1) = \int_{-1}^1 w(\xi) L_n(\xi) d\xi = \int_{-1}^1 w(\xi) L_n(\xi) L_0(\xi) d\xi = 0.$$

Next, we show that ϕ_{n+1} has exactly $n - 1$ roots in the open interval $(-1, 1)$. By (2.14) and (2.15), we have for $m \leq n$,

$$\begin{aligned} \int_{-1}^1 \beta \phi'_{n+1}(\xi) \phi'_m(\xi) d\xi &= - \int_{-1}^1 \phi_{n+1}(\xi) (\beta \phi'_m)'(\xi) d\xi \\ &= - \int_{-1}^1 \phi_{n+1}(\xi) L'_{m-1}(\xi) d\xi = 0. \end{aligned}$$

Since $L'_{m-1} \in \mathbb{P}_{m-2}$. Hence,

$$\int_{-1}^1 \phi_{n+1}(\xi) v(\xi) d\xi = 0, \quad \forall v \in \mathbb{P}_{n-2}. \quad (2.17)$$

In particular, choosing $v = 1$ we have

$$\int_{-1}^1 \phi_{n+1}(\xi) d\xi = 0.$$

Since ϕ_{n+1} is continuous, and its average over $(-1, 1)$ is zero, then it must change signs at least once in $(-1, 1)$. Let $\xi_1, \xi_2, \dots, \xi_k$ be all such points in

$(-1, 1)$ at which ϕ_{n+1} changes sign.

Suppose $k < n - 1$. We choose $v(\xi) = (\xi - \xi_1)(\xi - \xi_2) \cdots (\xi - \xi_k) \in \mathbb{P}_{n-2}$. By the orthogonality (2.17), we have

$$\int_{-1}^1 \phi_{n+1}(\xi) v(\xi) d\xi = 0. \quad (2.18)$$

However, the function $\phi_{n+1}(\xi) v(\xi)$ cannot change signs. This contradicts (2.18).

Suppose $k > n - 1$. Then without loss of generality, we assume $-1 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$ partitions $[-1, 1]$ into $k + 1$ small intervals, and assume the interface point $\hat{\alpha}$ is contained in the interval $\hat{\alpha} \in (\xi_i, \xi_{i+1})$. On all of k noninterface intervals, applying the standard Rolle's theorem, we conclude that the derivative of $\phi_{n+1}(\xi)$ has at least one zero in each of them. This means the weighted derivative $L_n(\xi) = \hat{\beta} \phi'_{n+1}(\xi)$ has at least k zeros on noninterface intervals.

On the interface interval (ξ_i, ξ_{i+1}) , since ϕ_{n+1} is not differentiable at the interior point $\hat{\alpha}$, we apply the generalized Rolle's theorem (Lemma 2.1). There exists a point c such that one of $\phi'_{n+1}(c-)$ and $\phi'_{n+1}(c+)$ is non-negative, and the other is non-positive. Then the following two quantities

$$L_n(c-) = \hat{\beta}(c-) \phi'_{n+1}(c-), \quad L_n(c+) = \hat{\beta}(c+) \phi'_{n+1}(c+)$$

should also satisfying this property, because $\hat{\beta}$ is strictly positive. Also note that L_n is a polynomial, and therefore continuous. Thus, $L_n(c) = 0$. This means $L_n(\xi)$ has another zero in the interface interval. Combining with the above result, $L_n(\xi)$ has at least $k + 1 > n$ zeros on $(-1, 1)$. This contradicts the first part of the theorem. \square

Next we show the consistence of the generalized orthogonal polynomials with standard orthogonal polynomials.

Lemma 2.2. *If the interface coincides with the boundary i.e., $\hat{\alpha} = \pm 1$, or if there is no jump of coefficient, i.e., $\beta^+ = \beta^-$, then $\{\phi_n\}$ and $\{L_n\}$ become standard Lobatto polynomial $\{\psi_n\}$ and Legendre polynomials $\{P_n\}$, respectively, up to a multiple constant.*

Proof. Suppose $\hat{\alpha} = -1$. The weight function $w(\xi) = (\beta^+)^{-1}$ becomes a constant. By the recurrence formula (2.10), it is easy to see that $L_n = c_n P_n$,

where c_n is a constant. By (2.13) we have

$$\phi_n(\xi) = \int_{-1}^{\xi} \frac{1}{\beta^+} L_{n-1}(s) ds = \frac{1}{\beta^+} c_{n-1} \int_{-1}^{\xi} P_{n-1}(s) ds = \frac{1}{\beta^+} c_{n-1} \psi_n(\xi),$$

for some constant c_{n-1} .

When $\hat{\alpha} = 1$, the argument is the same. When $\beta^+ = \beta^-$, the weight function $w(\xi) = (\beta^-)^{-1}$ becomes a constant. The corresponding result can be obtained following a similar argument as above. \square

3. Superconvergence Analysis

In this section, we investigate the superconvergence properties for IFE methods.

Recall that $\{\psi_n\}$ and $\{\phi_n\}$ are standard and generalized Lobatto polynomials on the reference element $\tau = [-1, 1]$, respectively. By the following affine mappings, we can obtain the local FE basis functions $\psi_{i,n}$ on each noninterface element τ_i and the IFE basic functions $\phi_{k,n}$ on the interface element τ_k .

$$\psi_{i,n}(x) := \psi_n(\xi) = \psi_n\left(\frac{2x - x_{i-1} - x_i}{h_i}\right), \quad n \geq 0. \quad (3.1)$$

$$\phi_{k,n}(x) := \phi_n(\xi) = \phi_n\left(\frac{2x - x_{k-1} - x_k}{h_k}\right), \quad n \geq 0. \quad (3.2)$$

Then the p -th degree local FE space $S_p^N(\tau_i)$ on noninterface elements τ_i , $i \neq k$, and IFE space $S_p^I(\tau_k)$ on interface element τ_k are defined by

$$S_p^N(\tau_i) = \text{span}\{\psi_{i,n} : n = 0, 1, \dots, p\}. \quad (3.3)$$

$$S_p^I(\tau_k) = \text{span}\{\phi_{k,n} : n = 0, 1, \dots, p\}. \quad (3.4)$$

Finally, the p -th degree global IFE space is defined by

$$S_p(\mathcal{T}_h) := \{v \in H_0^1(\Omega) : v|_{\tau_i} \in S_p^N(\tau_i), \ i \neq k, \text{ and } v|_{\tau_k} \in S_p^I(\tau_k)\}. \quad (3.5)$$

The IFE method for the interface problem (2.1) - (2.4) is: find $u_h \in S_p(\mathcal{T}_h)$ such that

$$(\beta u_h', v_h') + (\gamma u_h', v_h) + (cu_h, v_h) = (f, v_h), \quad \forall v_h \in S_p(\mathcal{T}_h). \quad (3.6)$$

3.1. IFE Interpolation

For superconvergence analysis, we define the following Sobolev spaces for $m \geq 1$ and $q \geq 1$

$$\tilde{W}_\beta^{m,q}(\Omega) = \left\{ v \in C(\Omega) : v|_{\Omega^\pm} \in W^{m,q}(\Omega^\pm), v|_{\partial\Omega} = 0, \right. \\ \left. \llbracket \beta v^{(j)}(\alpha) \rrbracket = 0, j = 1, 2, \dots, m \right\}, \quad (3.7)$$

equipped with the norm and semi-norm

$$\|v\|_{m,q}^2 = \|v\|_{m,q,\Omega^-}^2 + \|v\|_{m,q,\Omega^+}^2, \quad |v|_{m,q}^2 = |v|_{m,q,\Omega^-}^2 + |v|_{m,q,\Omega^+}^2.$$

For any $u \in \tilde{W}_\beta^{m,q}(\Omega)$, $m \geq 1$, we have the following Lobatto expansion of u on noninterface elements τ_i

$$u(x)|_{\tau_i} = \sum_{n=0}^{\infty} u_{i,n} \psi_{i,n}(x), \quad (3.8)$$

where

$$u_{i,0} = u(x_{i-1}), \quad u_{i,1} = u(x_i), \quad u_{i,n} = \frac{\int_{\tau_i} u'(x) \psi'_{i,n}(x) dx}{\int_{\tau_i} \psi'_{i,n}(x) \psi'_{i,n}(x) dx}, \quad n \geq 2. \quad (3.9)$$

On the interface element τ_k , since the flux $\beta u'$ is continuous, then it can be expanded by generalized Legendre polynomials $\{L_{k,n}\}$

$$\beta u'(x) = \sum_{n=0}^{\infty} u_{k,n} L_{k,n}(x).$$

Dividing by β and then integrating on both sides yield the expansion for u :

$$u(x)|_{\tau_k} = \sum_{n=0}^{\infty} u_{k,n} \int_{x_{k-1}}^x \frac{1}{\beta(x)} L_{k,n}(x) dx = \sum_{n=0}^{\infty} u_{k,n} \phi_{k,n}(x). \quad (3.10)$$

By the orthogonality (2.15) and the properties of generalized Lobatto polynomials in Theorem 2.2, we have

$$u_{k,0} = u(x_{k-1}), \quad u_{k,1} = u(x_k), \quad u_{k,n} = \frac{\langle u, \phi_{k,n} \rangle_{\tau_k}}{\langle \phi_{k,n}, \phi_{k,n} \rangle_{\tau_k}}, \quad n \geq 2, \quad (3.11)$$

where

$$\langle u, v \rangle_{\tau_k} = \int_{x_{k-1}}^{x_k} \beta u'(x) v'(x) dx, \quad \forall u, v \in \tilde{W}_\beta^{m,q}(\Omega).$$

Using the (generalized) Lobatto expansions (3.8) and (3.10) on noninterface and interface elements, we define the IFE interpolation $\mathcal{I}_h : \tilde{W}_\beta^{m,q}(\Omega) \rightarrow S_p(\mathcal{T}_h)$ as follows

$$(\mathcal{I}_h u)|_{\tau_i} = \begin{cases} \sum_{n=0}^p u_{i,n} \psi_{i,n}(x), & \text{if } i \neq k \\ \sum_{n=0}^p u_{i,n} \phi_{i,n}(x), & \text{if } i = k. \end{cases} \quad (3.12)$$

The IFE interpolation $\mathcal{I}_h u$ plays an important role in our superconvergence analysis. To study the approximation properties of $\mathcal{I}_h u$, we need to define a class of integral operators $D_x^{-j}, j \geq 1$ on any function $v \in \tilde{W}_\beta^{m,q}(\Omega)$ as

$$(D_x^{-1}v)|_{\tau_i} = \int_{x_{i-1}}^x v(x) dx, \quad (D_x^{-j}v)|_{\tau_i} = \int_{x_{i-1}}^x D_x^{-(j-1)}v(x) dx, \quad j > 1.$$

Lemma 3.1. *There holds for all $j \leq n-2$*

$$D_x^{-j} \phi_{k,n}(x_{k-1}) = D_x^{-j} \phi_{k,n}(x_k) = 0. \quad (3.13)$$

Proof. Note from (2.17) and Theorem 2.2 that

$$\phi_{k,n}(x_{k-1}) = \phi_{k,n}(x_k) = 0, \quad \int_{x_{k-1}}^{x_k} \phi_{k,n}(x) v(x) dx = 0, \quad \forall v \in \mathbb{P}_{n-3}(\tau_k). \quad (3.14)$$

Choosing $v = 1$ in the above equation, we immediately obtain

$$D_x^{-1} \phi_{k,n}(x_k) = \int_{x_{k-1}}^{x_k} \phi_{k,n}(x) dx = 0 = D_x^{-1} \phi_{k,n}(x_{k-1}), \quad \forall n \geq 3.$$

Moreover, noticing that $D_x^{-1}v \in \mathbb{P}_{n-3}(\tau_k)$ for all $v \in \mathbb{P}_{n-4}(\tau_k)$, we have, from (3.14) and the integration by parts,

$$\int_{\tau_k} D_x^{-1} \phi_{k,n}(x) v(x) dx = - \int_{\tau_k} \phi_{k,n}(x) D_x^{-1} v(x) dx = 0, \quad \forall v \in \mathbb{P}_{n-4}(\tau_k).$$

In other words, $D_x^{-1} \phi_{k,n}$ shares the same properties of $\phi_{k,n}$, *i.e.*,

$$D_x^{-1} \phi_{k,n}(x_k) = D_x^{-1} \phi_{k,n}(x_{k-1}) = 0, \quad \int_{x_{k-1}}^{x_k} D_x^{-1} \phi_{k,n}(x) v(x) dx = 0, \quad v \in \mathbb{P}_{n-4}(\tau_k).$$

By recursion, there holds for all $j \leq n - 3$

$$D_x^{-j} \phi_{k,n}(x_k) = D_x^{-j} \phi_{k,n}(x_{k-1}) = 0, \quad \int_{x_{k-1}}^{x_k} D_x^{-j} \phi_{k,n}(x) v(x) dx, \quad v \in \mathbb{P}_{n-3-j}(\tau_k),$$

which yields

$$D_x^{-(j+1)} \phi_{k,n}(x_k) = \int_{x_{k-1}}^{x_k} D_x^{-j} \phi_{k,n}(x) dx = 0 = D_x^{-(j+1)} \phi_{k,n}(x_{k-1}), \quad j \leq n-3.$$

This finishes our proof. \square

Now we are ready to show the approximation properties of the IFE interpolation $\mathcal{I}_h u$.

Lemma 3.2. *Assume that $u \in \tilde{W}_\beta^{p+2,\infty}(\Omega)$, and $\mathcal{I}_h u$ is the IFE interpolation of u defined by (3.12). The following orthogonality and approximation properties hold true.*

1. *Orthogonality:*

$$\int_{\tau_i} \beta(u - \mathcal{I}_h u)' v' dx = 0, \quad \forall v \in S_p(\mathcal{T}_h), \quad i = 1, \dots, N. \quad (3.15)$$

2. *For $p \geq 2$, the function value of $\mathcal{I}_h u$ is superconvergent at the interior roots of the (generalized) Lobatto polynomials $\psi_{i,p+1}$ on noninterface elements or $\phi_{k,p+1}$ on interface element, i.e.,*

$$|(u - \mathcal{I}_h u)(l_{im})| \leq Ch^{p+2} |u|_{p+2,\infty,\tau_i}, \quad i = 1, 2, \dots, N, \quad (3.16)$$

where l_{im} , $m = 1, \dots, p-1$ are interior roots of $\psi_{i,p+1}$ or $\phi_{i,p+1}$ on τ_i .

3. *The derived function value of $\mathcal{I}_h u$ is superconvergent at roots of the Legendre polynomial $P_{i,p}$ on noninterface elements τ_i , $i \neq k$, i.e.,*

$$|(u' - (\mathcal{I}_h u)')(g_{in})| \leq Ch^{p+1} |u|_{p+2,\infty,\tau_i}, \quad i \neq k, \quad (3.17)$$

where g_{in} , $n = 1, \dots, p$ are roots of $P_{i,p}$ on τ_i .

4. *The flux value is superconvergent at roots of the generalized Legendre polynomial $L_{k,p}$ on the interface element τ_k , i.e.,*

$$|(\beta u' - (\beta \mathcal{I}_h u)')(g_{kn})| \leq Ch^{p+1} |u|_{p+2,\infty,\tau_k}, \quad (3.18)$$

where g_{kn} , $n = 1, \dots, p$ are roots of $L_{k,p}$ on τ_k .

Proof. By (3.8), (3.10) and (3.12), we have

$$(u - \mathcal{I}_h u)|_{\tau_i} = \begin{cases} \sum_{n=p+1}^{\infty} u_{i,n} \psi_{i,n}(x), & \text{if } i \neq k, \\ \sum_{n=p+1}^{\infty} u_{i,n} \phi_{i,n}(x), & \text{if } i = k. \end{cases} \quad (3.19)$$

Then (3.15) follows from the orthogonal properties of (generalized) Lobatto polynomials.

On each noninterface element $\tau_i, i \neq k$, we have from (3.9)

$$\begin{aligned} u_{i,n} &= \frac{h_i}{2n-1} \int_{\tau_i} u'(x) \psi'_{i,n}(x) dx = \frac{2}{2n-1} \int_{-1}^1 \frac{du(\xi)}{d\xi} P_{n-1}(\xi) d\xi \\ &= \frac{2}{2n-1} C_n \int_{-1}^1 \frac{du(\xi)}{d\xi} \frac{d^{n-1}}{d\xi^{n-1}} (1-\xi^2)^{n-1} d\xi \\ &= \frac{2}{2n-1} C_{nj} \int_{-1}^1 \frac{d^j u(\xi)}{d\xi^j} \frac{d^{n-j}}{d\xi^{n-j}} (1-\xi^2)^{n-1} d\xi, \quad j \leq n. \end{aligned}$$

Here, C_{nj} is a constant depending on n and j . Since

$$\frac{d^j u(\xi)}{d\xi^j} = \left(\frac{h_i}{2}\right)^j \frac{d^j u(x)}{dx^j},$$

then

$$|u_{i,n}| \leq Ch^j |u|_{j,\infty,\tau_i}, \quad j \leq n. \quad (3.20)$$

On the interface element τ_k , by (3.11),

$$\begin{aligned} u_{k,n} &= \frac{1}{\langle \phi_n, \phi_n \rangle_{\tau}} \frac{h_k}{2} \int_{\tau_k} (\beta u')(x) \phi_{k,n}(x) dx \\ &= \frac{(-1)^{j-1}}{\langle \phi_n, \phi_n \rangle_{\tau}} \frac{h_k}{2} \int_{x_{k-1}}^{x_k} (\beta u')^{(j+1)}(x) D_x^{-j} \phi_{k,n}(x) dx, \quad j \leq n-2. \end{aligned}$$

Here in the last step, we have used the integration by parts and (3.13). Since $\|D_x^{-1} v\|_{L^\infty(\tau_k)} \leq h \|v\|_{L^\infty(\tau_k)}$, we have

$$|u_{k,n}| \leq Ch^{j+2} |\beta u^{(j+2)}|_{L^\infty(\tau_k)}, \quad j \leq n-2. \quad (3.21)$$

Then (3.16) follows from (3.19)-(3.21).

Also note that

$$\begin{aligned}(u' - (\mathcal{I}_h u)')|_{\tau_i} &= \frac{2}{h_i} \sum_{n=p}^{\infty} u_{i,n} P_{i,n}(x), \quad \text{if } i \neq k, \\ (\beta u' - (\beta \mathcal{I}_h u)')|_{\tau_k} &= \frac{2}{h_k} \sum_{n=p}^{\infty} u_{k,n} L_{k,n}(x).\end{aligned}$$

Then (3.17)-(3.18) follow from (3.20)-(3.21). The proof is complete. \square

3.2. Superconvergence for diffusion interface problems

We first consider the diffusion interface problem, *i.e.*, $\gamma = c = 0$ in (2.1). Assume that $u_h \in S_p(\mathcal{T}_h)$ is the IFE solution of

$$a(u_h, v_h) := (\beta u_h', v_h') = (f, v_h), \quad \forall v_h \in S_p(\mathcal{T}_h). \quad (3.22)$$

By the Poincaré inequality, and the orthogonality (3.15), we have

$$\begin{aligned}\|\mathcal{I}_h u - u_h\|_1^2 &\leq C |\mathcal{I}_h u - u_h|_1^2 \leq Ca(\mathcal{I}_h u - u_h, \mathcal{I}_h u - u_h) \\ &= Ca(\mathcal{I}_h u - u, \mathcal{I}_h u - u_h) = 0.\end{aligned}$$

Hence, $u_h = \mathcal{I}_h u$. That means u_h inherits all superconvergent properties (3.16) - (3.18) of $\mathcal{I}_h u$. We summarize these results in the following theorem.

Theorem 3.1. *Let $\mathcal{T}_h = \{\tau_i\}_{i=1}^N$ be a mesh of Ω such that the interface $\alpha \in \tau_k$. Let $u_h \in S_p(\mathcal{T}_h)$ be the IFE solution of (3.22) where $p \geq 2$, and $u \in \tilde{W}_\beta^{p+2,\infty}(\Omega)$ be the exact solution of (2.1) - (2.4). Then we have the following results.*

- u_h is exact at the mesh points, *i.e.*,

$$(u - u_h)(x_i) = 0, \quad \forall i = 0, 1, \dots, N. \quad (3.23)$$

- On every noninterface element τ_i , $i \neq k$, u_h is superconvergent at roots of Lobatto polynomial $\psi_{i,p+1}$, and the derivative u_h' is superconvergent at roots of Legendre polynomial $P_{i,p}$, *i.e.*,

$$(u - u_h)(l_{im}) = O(h^{p+2}), \quad (u' - u_h')(g_{in}) = O(h^{p+1}). \quad (3.24)$$

- On the interface element τ_k , u_h is superconvergent at roots of generalized Lobatto polynomial $\phi_{k,p+1}$, and the flux $\beta u_h'$ is superconvergent at roots of generalized Legendre polynomial $L_{k,p}$, *i.e.*,

$$(u - u_h)(l_{km}) = O(h^{p+2}), \quad (\beta u' - \beta u_h')(g_{kn}) = O(h^{p+1}). \quad (3.25)$$

3.3. Superconvergence for general elliptic interface problems

We consider the general second-order elliptic interface problem. As the standard finite element approximation, we cannot expect u_h is exact at the mesh points. However, using a Green's function, we may establish the following superconvergence result.

Theorem 3.2. *Let $\mathcal{T}_h = \{\tau_i\}_{i=1}^N$ be an partition of Ω such that the interface $\alpha \in \tau_k$. Let $u_h \in S_p(\mathcal{T}_h)$ be the IFE solution of (3.6) where $p \geq 2$, and $u \in \tilde{W}_\beta^{p+2,\infty}(\Omega)$ be the exact solution of (2.1) - (2.4). Then we have the following superconvergence results.*

- At the mesh points x_i , $i = 0, 1, \dots, N$. i.e.,

$$(u - u_h)(x_i) = O(h^{2p}). \quad (3.26)$$

- On every noninterface element τ_i , $i \neq k$, u_h is superconvergent at roots $\psi_{i,p+1}$, and u'_h is superconvergent at roots of $P_{i,p}$, i.e.,

$$(u - u_h)(l_{im}) = O(h^{p+2}), \quad (u' - u'_h)(g_{in}) = O(h^{p+1}). \quad (3.27)$$

- On interface element τ_k , u_h is superconvergent at roots of $\phi_{k,p+1}$ when $p \geq 2$, and $\beta u'_h$ is superconvergent at roots of $L_{k,p}$, i.e.,

$$(u - u_h)(l_{km}) = O(h^{p+2}), \quad (\beta u' - \beta u'_h)(g_{kn}) = O(h^{p+1}). \quad (3.28)$$

Proof. For any given point z , we denote by $G(\cdot, z)$ the Green's function of (2.1) associated with z , that is,

$$\begin{aligned} -(\beta G'(\cdot, z))' - \gamma G'(\cdot, z) + cG(\cdot, z) &= 0, \quad x \in \Omega^- \cup \Omega^+, \quad x \neq z, \\ G(a, z) = G(b, z) &= 0, \quad [G'(z, z)] = 1, \quad [G(z, z)] = 0, \end{aligned}$$

where $[G(z, z)] = G(z^+, z) - G(z^-, z)$ denotes the jump of $G(\cdot, z)$ at the point z . By the property of Green's function, $G(\cdot, z) \in \tilde{H}_\beta^{p+1}$ has $(p+1)$ -th order smooth derivative in intervals $[a, z)$ and $(z, b]$, respectively. Then for all $v_h \in S_p(\mathcal{T}_h)$, we have

$$\begin{aligned} (u - u_h)(x_i) &= a(u - u_h, G(\cdot, x_i)) \\ &= a(u - u_h, G(\cdot, x_i) - v_h) \\ &\leq \|u - u_h\|_1 \inf_{v_h \in S_p(\mathcal{T}_h)} \|G(\cdot, x_i) - v_h\|_1 \\ &\leq Ch^{2k} \|u\|_{k+1}. \end{aligned} \quad (3.29)$$

We proved (3.26). Next, we let $g_h(\cdot, z)$ be the discrete Green's function associated with the point z such that

$$a(v_h, g_h(\cdot, z)) = v_h(z), \quad \forall v_h \in S_p(\mathcal{T}_h). \quad (3.30)$$

Let $g_I = \mathcal{I}_h G(\cdot, z) \in S_p(\mathcal{T}_h)$ be the interpolation of $G(\cdot, z)$, and τ_0 be the interval containing the point z . Then

$$\begin{aligned} \|G(\cdot, z) - g_I\|_1 &= \|G(\cdot, z) - g_I\|_{1, \Omega \setminus \tau_0} + \|G(\cdot, z) - g_I\|_{1, \tau_0} \\ &\leq Ch + C\|G(\cdot, z)\|_{1, \tau_0} \leq Ch. \end{aligned}$$

Since $g_h(\cdot, z)$ is the IFE approximation of the Green's function $G(\cdot, z)$, we get

$$\|G(\cdot, z) - g_h(\cdot, z)\|_1 \leq C \inf_{v_h \in S_p(\mathcal{T}_h)} \|G(\cdot, z) - v_h\|_1 \leq Ch.$$

Then

$$\begin{aligned} \|g_h(\cdot, z)\|_{2,1} &\leq \|g_h(\cdot, z) - g_I\|_{2,1} + \|g_I\|_{2,1, \Omega \setminus \tau_0} + \|g_I\|_{2,1, \tau_0} \\ &\leq Ch^{-1}\|g_h(\cdot, z) - g_I\|_{1,1} + \|G(\cdot, z)\|_{2,1, \Omega \setminus \tau_0} + Ch^{-1}\|g_I\|_{1,1, \tau_0} \leq C, \end{aligned}$$

where in the second inequality, we have used the inverse inequality. Then by choosing $v = u_h - \mathcal{I}_h u$ in (3.30) and using the orthogonality,

$$\begin{aligned} &(u_h - \mathcal{I}_h u)(z) \\ &= a(u - \mathcal{I}_h u, g_h(z, \cdot)) \\ &= - \int_{\Omega} \gamma(u - \mathcal{I}_h u) g'_h + \int_{\Omega} c(u - \mathcal{I}_h u) g_h \\ &= \int_{\Omega \setminus \tau_k} \gamma D_x^{-1}(u - \mathcal{I}_h u) g''_h - \int_{\tau_k} \gamma(u - \mathcal{I}_h u) g'_h + \int_{\Omega} c D_x^{-1}(u - \mathcal{I}_h u) g'_h \\ &\leq Ch \|u - \mathcal{I}_h u\|_{L^\infty} \|g_h\|_{2,1}. \end{aligned}$$

Here in the last step, we have used the integration by parts, the continuity of g'_h in all the noninterface elements $\Omega \setminus \tau_k$, and

$$D_x^{-1}(u - \mathcal{I}_h u)(x_i) = D_x^{-1}(u - \mathcal{I}_h u)(x_{i-1}) = 0, \quad i = 1, \dots, N.$$

Consequently,

$$\|u_h - \mathcal{I}_h u\|_{L^\infty} \leq Ch^{p+2} \|u\|_{p+2, \infty}.$$

By the inverse inequality,

$$\|(u_h - \mathcal{I}_h u)'\|_{L^\infty} \leq h^{-1} \|u_h - \mathcal{I}_h u\|_{L^\infty} \leq Ch^{p+1} \|u\|_{p+2, \infty}.$$

Then (3.27) and (3.28) follow from (3.16). \square

Remark 3.1. *The assumption $p \geq 2$ in Theorems 4.1 and 4.2 rules out the linear element case $p = 1$ only because the convergent rate $O(h^{p+2})$ at the Lobatto (generalized Lobatto) points are not valid for linear element, since the best rate is h^{2p} , which is h^2 in the linear case. All other results are trivially extended to the linear element case.*

Remark 3.2. *In general, there is no superconvergence at the interface point, because the IFE method treats the interface as an interior point. Even if there is no coefficient jump, the IFE method (becomes standard FE method) has no superconvergence behavior at a random interior point, unless it coincides with Lobatto or Gauss points.*

4. Numerical Experiments

In this section, we present some numerical experiments to demonstrate the superconvergence of IFE methods.

We use the following example as the exact solution for the interface problem

$$u(x) = \begin{cases} \frac{1}{\beta^-} \cos(x), & \text{if } x \in [0, \alpha), \\ \frac{1}{\beta^+} \cos(x) + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+} \right) \cos(\alpha), & \text{if } x \in (\alpha, 1]. \end{cases} \quad (4.1)$$

It is easy to verify that

$$\llbracket u(\alpha) \rrbracket = 0, \quad \llbracket \beta u^{(j)}(\alpha) \rrbracket = 0, \quad \forall j \geq 1.$$

We compute the error $e_h = u_h - u$ in the following norms

$$\begin{aligned} \|e_h\|_N &= \max_{x \in \{x_i\}} |u_h(x) - u(x)|, \quad \|e_h\|_{L^\infty} = \max_{x \in \Omega} |u_h(x) - u(x)|, \\ \|e_h\|_L &= \max_{x \in \{l_{ip}\}} |u_h(x) - u(x)|, \quad \|\beta e_h'\|_G = \max_{x \in \{g_{ip}\}} |\beta u_h'(x) - \beta u'(x)|, \\ \|e_h\|_{L^2} &= \left(\int_{\Omega} |u_h - u|^2 dx \right)^{\frac{1}{2}}, \quad |e_h|_{H^1} = \left(\int_{\Omega} |u_h' - u'|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Here, $\|e_h\|_N$ denotes the maximum error over all the mesh points. $\|e_h\|_{L^\infty}$ is the infinity norm over the whole domain Ω . To compute it, we select 10 uniformly distributed points on each non-interface element, and select 10 points in each sub-element of an interface element. Among all these sample points, we compute the largest discrepancy from the exact solution. $\|\beta e_h'\|_G$

is the maximum error of flux over all generalize Legendre points. $\|e_h\|_L$ is maximum errors over all (generalized) Lobatto points, respectively. $\|e_h\|_{L^2}$ and $|e_h|_{H^1}$ are the standard Sobolev L^2 - and semi- H^1 - norms.

In our computation, we start from a uniform partition consisting of 10 elements. Due to the finite machine precision, we choose different sets of meshes for different polynomial degrees. The convergence rate is calculated by using linear regression of the errors.

We consider the general elliptic interface problem, and choose the coefficient $(\beta^-, \beta^+) = (1, 10)$, $\gamma = 1$, $c = 10$, and the interface $\alpha = \pi/6$. We test the IFE approximation of degree $p = 1, 2, 3$, and errors in the aforementioned norms are reported in Tables 1, 2, and 3, respectively. We can observe that the IFE solutions u_h have a superconvergence rate of $O(h^{2p})$ at mesh points, compared to the rate $O(h^{p+1})$ in the infinity norm $\|\cdot\|_{L^\infty}$. At (generalized) Legendre-Gauss points and (generalized) Lobatto points ($p = 2, 3$), the convergence rates are $O(h^{p+1})$ and $O(h^{p+2})$, respectively. These data indicate that at these special points, IFE solution are super-close to the exact solution, and the convergence rates are one order higher than optimal rate. Moreover, the convergence rates are $O(h^{p+1})$ and $O(h^p)$ in $\|\cdot\|_{L^2}$ and $|\cdot|_{H^1}$ norm, which is consistent with the diffusion interface problem in [2].

$1/h$	$\ e_h\ _N$	$\ e_h\ _{L^\infty}$	$\ \beta e'_h\ _G$	$\ e_h\ _{L^2}$	$\ e_h\ _{H^1}$
10	2.1744e-04	1.1746e-03	1.3315e-03	5.2782e-04	1.9765e-02
20	5.4176e-05	3.0041e-04	3.6827e-04	1.3133e-04	9.8741e-03
40	1.3887e-05	7.6075e-05	9.7454e-05	3.3172e-05	4.9943e-03
80	3.4724e-06	1.9151e-05	2.4963e-05	8.2948e-06	2.4975e-03
160	8.6897e-07	4.8050e-06	6.3178e-06	2.0753e-06	1.2492e-03
320	2.1746e-07	1.2034e-06	1.5896e-06	5.1941e-07	6.2496e-04
640	5.4437e-08	3.0113e-07	3.9878e-07	1.2998e-07	3.1275e-04
1280	1.3613e-08	7.5317e-08	9.9862e-08	3.2492e-08	1.5637e-04
rate	1.9944	1.9911	1.9639	1.9977	0.9973

Table 1: Error of P_1 IFE Solution with $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

Next we illustrate superconvergence behavior at roots of (generalized) orthogonal polynomials. In Figures 2, 3, and 4, we list the plots of solution error $u_h - u$ and the flux error $\beta u'_h - \beta u'$ on the mesh consisting of 10 elements. Also, we highlight the roots of corresponding orthogonal polynomials by star with red color. Clearly, at those points, errors are much smaller compared to other points. Note that the interface $\alpha \in (0.5, 0.6)$, and the red-color-marked points on this interface element are roots of generalized Lobatto/Legendre polynomials.

$1/h$	$\ e_h\ _N$	$\ e_h\ _{L^\infty}$	$\ e_h\ _L$	$\ \beta e'_h\ _G$	$\ e_h\ _{L^2}$	$\ e_h\ _{H^1}$
10	4.6588e-08	3.4957e-06	1.1609e-07	5.2607e-06	1.1836e-06	7.8010e-05
20	3.5220e-09	4.5800e-07	6.6222e-09	6.4788e-07	1.5041e-07	1.9653e-05
30	6.0039e-10	1.3810e-07	1.5622e-09	1.9227e-07	4.4385e-08	8.6656e-06
40	1.7464e-10	5.8732e-08	5.4228e-10	8.1045e-08	1.9287e-08	5.0173e-06
50	7.7451e-11	3.1323e-08	2.2667e-10	4.1847e-08	9.9480e-09	3.2290e-06
60	4.4824e-11	1.8047e-08	8.9915e-11	2.6163e-08	5.7351e-09	2.2373e-06
70	2.1843e-11	1.1344e-08	5.3558e-11	1.5137e-08	3.5913e-09	1.6314e-06
80	1.1342e-11	7.5906e-09	3.5621e-11	1.0142e-08	2.4178e-09	1.2565e-06
rate	3.9848	2.9453	3.9009	2.9927	2.9760	1.9828

Table 2: Error of P_2 IFE Solution with $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

$1/h$	$\ e_h\ _N$	$\ e_h\ _{L^\infty}$	$\ e_h\ _L$	$\ \beta e'_h\ _G$	$\ e_h\ _{L^2}$	$\ e_h\ _{H^1}$
10	1.4783e-10	4.8213e-08	6.4054e-10	2.8803e-08	2.2669e-08	2.1613e-06
12	5.5940e-11	2.3269e-08	2.5075e-10	1.3966e-08	1.0933e-08	1.2487e-06
14	2.0409e-11	1.2565e-08	1.1383e-10	7.5625e-09	5.8994e-09	7.8535e-07
16	6.4971e-12	7.3669e-09	5.7553e-11	4.4424e-09	3.4572e-09	5.2588e-07
18	1.1036e-12	4.5997e-09	3.1582e-11	2.7773e-09	2.1586e-09	3.6945e-07
20	1.1257e-12	3.0181e-09	1.8479e-11	1.8243e-09	1.4172e-09	2.6956e-07
22	1.7312e-12	2.0615e-09	1.1387e-11	1.2472e-09	9.6900e-10	2.0276e-07
24	1.7318e-12	1.4556e-09	7.3233e-12	8.8119e-10	6.8511e-10	1.5636e-07
rate	5.8768	3.9983	5.1058	3.9837	3.9977	3.0000

Table 3: Error of P_3 IFE Solution with $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

The numerical results for diffusion interface problem are similar, except at mesh points there will be only roundoff error. We have also conducted numerical experiments for different configuration of interface locations α , and different sets of coefficients β^\pm , including large coefficient contrast. Similar superconvergence properties have been observed as the exemplified example, hence we omit these data in the article.

5. Conclusion

In this article, we develop explicitly, the orthogonal IFE basis functions. First we construct a set of bases for flux, then integrate to obtain basis functions for the primary unknown. The procedure is somewhat “reversed” from the classical approach in constructing IFE basis functions. The superconvergence behavior has been observed and proved for general elliptic interface problems in the one dimensional setting. At the roots of generalized Lobatto

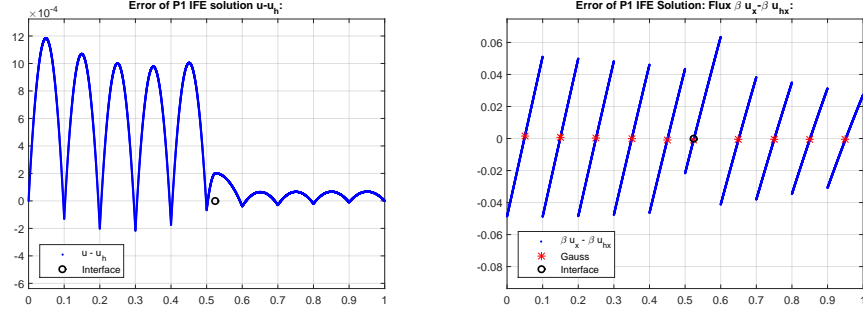


Figure 2: Error and flux error of P_1 IFE solution. $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

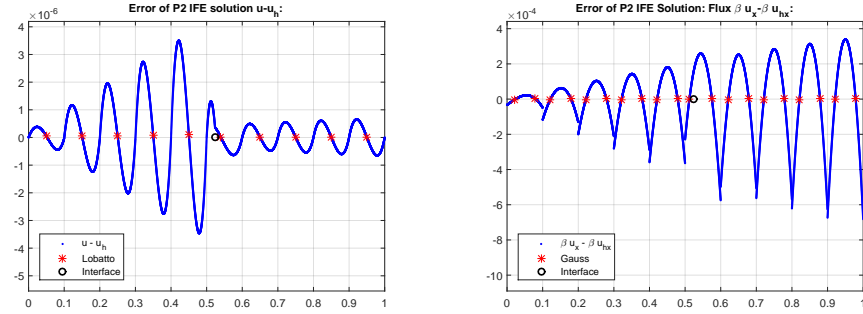


Figure 3: Error and flux error of P_2 IFE solution. $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

polynomial of degree $p + 1$, the IFE solution is superconvergent to the exact solution with order $p + 2$ (comparing with the optimal order $p + 1$); at the roots of generalized Legendre polynomial of degree p , the derivative of the IFE solution is superconvergent to the derivative of the exact solution with order $p + 1$ (comparing with the optimal order p). In addition, the convergent rate at all mesh points (including those of the interface element) is of order $2p$ (comparing with the optimal order $p + 1$). The idea presented in this article seems extendable to the two dimensional elliptic interface problems (at least for the tensor-product space case), which will be an interesting future work.

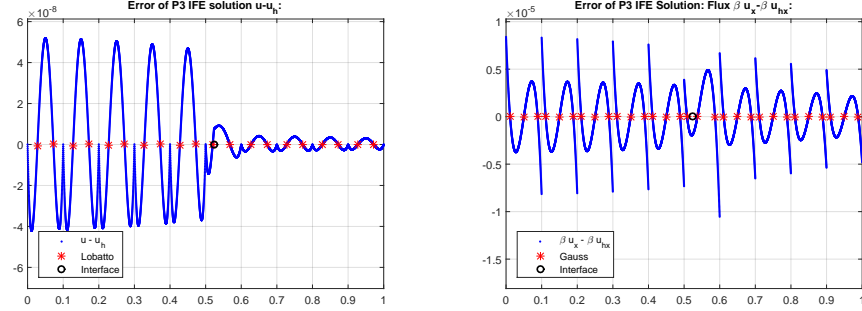


Figure 4: Error and flux error of P_3 IFE solution. $\beta^- = 1$, $\beta^+ = 10$, $\alpha = \pi/6$, $\gamma = 1$, $c = 10$.

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